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Technical Note T.N. 13

Asymptotic expansion of Stirling numbers

by

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The following problem was posed by the Statistical Department. Let Stirling's number C_n^j be defined by

$$x(x+1)(x+2)\dots(x+n-1) = \sum_0^{n-1} C_n^j\, x^{n-j} \qquad (1)$$

then an asymptotic expansion of C_n^j is sought where j is moderate and n very large.

This problem may be solved in the following way. For (1) we may write

$$\frac{\Gamma(x+n)}{x^n\Gamma(x)} = \sum_0^{n-1} C_n^j\, x^{-j} . \qquad (2)$$

If x is very large, x >> n we have (see e.g. Whittaker and Watson, Modern Analysis 13.6)

$$\log \Gamma(x+n) = (x+n-\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log(2\pi) + \sum_{m=1} \frac{(-1)^{m-1}\phi_{m+1}(n)}{m(m+1)x^m} , \qquad (3)$$

where $\phi_m(z)$ is the m^{th} Bernoullian polynomial which is defined by

$$\frac{t\, e^{zt}}{e^t-1} = \sum_0^{\infty} \phi_m(z)\, \frac{t^m}{m!} . \qquad (4)$$

We note that

$$\phi_o = 1 \qquad \phi_1 = z-\tfrac{1}{2} \qquad \phi_2 = z^2-z+\frac{1}{6}$$

$$\phi_3 = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z , \quad \text{and generally}$$

$$\phi_m(z) = (B+z)^{(m)} \qquad (5)$$

where

$$B_o = 1,\quad B_1 = -\frac{1}{2} ,\quad B_2 = \frac{1}{6} ,\quad B_3 = 0,\quad B_4 = -\frac{1}{30} ,\quad B_5 = 0 \dots .$$

From (3) it follows that

$$\log \frac{\Gamma(x+n)}{x^n\Gamma(x)} = \sum_{m=1} \frac{\Psi_m(n)}{x^m} , \qquad (6)$$

where

$$\Psi_m(z) = (-1)^{m-1} \left\{ \frac{\phi_{m+1}(z)-\phi_{m+1}(0)}{m(m+1)} \right\} . \qquad (7)$$

We note that

$$\Psi_1 = \frac{1}{2}(z^2-z) \qquad = \frac{1}{2}z(z-1)$$

$$\Psi_2 = \frac{1}{6}(z^3 - \frac{3}{2}z^2 + \frac{1}{2}z) \quad = -\frac{1}{6}z(z-1)(2z-1)$$

and symbolically

$$\psi_m(z)=(-1)^{m+1}\left\{\frac{(z+B)^{(m+1)}-B^{(m+1)}}{m(m+1)}\right\}$$

$$=(-1)^{m+1}\left\{\frac{z^{m+1}}{m(m+1)}-\frac{z^m}{2m}+\frac{z^{m-1}}{12}+\ldots\right\} \qquad (8)$$

From (2) and (6) we may derive that

$$\sum_{o}^{n-1} C_n^j\, x^{-j} = \exp \sum_{m=1} \frac{\psi_m(n)}{x^m}$$

$$= \prod_{m=1}\left\{\exp\frac{\psi_m(n)}{x^m}\right\} \qquad (9)$$

or written out in full changing x in x^{-1}

$$\sum_{o}^{n-1} C_n^j\, x^{+j} = \prod_{m=1}\left\{1+\frac{\psi_m\, x^m}{1!}+\frac{\psi_m^2 x^{2m}}{2!}+\frac{\psi_m^3 x^{3m}}{3!}+\ldots\right\}. \qquad (10)$$

Comparing the coefficients of x^j on both sides gives

$$C_n^j = \sum_{k_1 k_2 k_3\ldots}\frac{\psi_1^{k_1}\psi_2^{k_2}\psi_3^{k_3}\ldots}{k_1!k_2!k_3!\ldots} \qquad (11)$$

where $k_1+2k_2+3k_3+\ldots=j$.

This important relation is obviously an identity which of course could have been proved in a more direct way.
The polynomial ψ_m is of the degree (m+1) in n. Therefore the terms on the right-hand side of (11) can be written in the following order

$$C_n^j = \frac{\psi_1^{\,j}}{j!}+\frac{\psi_1^{\,j-2}\psi_2}{(j-2)!}+\left\{\frac{\psi_1^{\,j-3}\psi_3}{(j-3)!}+\frac{\psi_1^{\,j-4}\psi_2^{\,2}}{(j-4)!2!}\right\}+\ldots \qquad . (12)$$

Hence we obtain the asymptotic expression

$$C_n^j \approx \frac{\{\frac{1}{2}n(n-1)\}^j}{j!} - \frac{\{\frac{1}{2}n(n-1)\}^{j-2}\{-\frac{1}{6}n(n-1)(2n-1)\}}{(j-2)!}$$

$$\approx \frac{\{\frac{1}{2}n(n-1)\}^j}{j!}\left\{1-\frac{2j(j-1)(2n-1)}{3n(n-1)}\right\}. \qquad (13)$$

This indicates that this expression only holds for $j \ll \sqrt{n}$.

For the Stirling numbers of the second specis as defined by

$$\frac{x^n}{x(x+1)\dots(x+n-1)} = \sum_{0}^{\infty} (-1)^j \, D_n^j \, x^{-j} \tag{14}$$

we obtain of course instead of (11)

$$D_n^j = (-1)^j \sum_{k_1 k_2 k_3 \dots} (-1)^{k_1+k_2+k_3\dots} \frac{\psi_1^{k_1} \psi_2^{k_2} \psi_3^{k_3} \dots}{k_1! k_2! k_3!} \tag{15}$$

so that (13) becomes

$$D_n^j \approx \frac{\{\frac{1}{2}n(n-1)\}^j}{j!} \left\{ 1 + \frac{2j(j-1)(2n-1)}{3n(n-1)} \right\} . \tag{16}$$